THE LOGICAL PLURALISM

§ 1

It is only now that those who enjoy researches in logic can remember Kant's assertion: with pleasure. Since Aristotle, logic has not taken any step ahead, but it has not been obliged to take any step backwards either. After about one century of intense research, the domain of formal logic has been so enlarged, its discoveries have been so numerous, the discriminations so subtle and the new ideas so rich, that if, indeed, formal logic has not been obliged to take any step backwards, it is beyond any doubt that several steps ahead have been taken. Logic has again become a living science, developing permanently; it stopped being isolated, keeping in touch with the other branches of positive science.

Nevertheless, one has to confess that the flourishing of logic research has had no echo in the consciousness of cultivated people. Even those who learnt that such researches are being carried out have the feeling that it is a matter of special chapters, which only professionals are interested in.

We think that there are two causes for this situation: on the one hand, the researches on formal logic have been expressed for the last hundred years in a symbolic language. Everyday language proved too ambiguous and, at the same time, much less flexible to express the various, but precise, shades of meaning used by a contemporary logician. No doubt that all sciences were obliged, in the moment of their creation, to build up a language of their own and, at the moment of reaching full maturity, this language was reduced to the mathematical language. The language of natural sciences is not much different from everyday speech; it is only the meaning of certain terms that has to be specified, the sentence preserving the structure of the natural language. The mathematical language of the sciences that have become mathematical is inaccessible to non-mathematicians, but the permanent contact between theoretical physics and mathematical physics has made it possible to translate several theories belonging to mathematical physics into a language which is not mathematical. On the contrary, the language of symbolic logic, bearing all the characteristics of a mathematical language, which makes the less informed reader reluctant in front of it, has, besides that, the disadvantage of being different from mathematical language and, therefore, it is at first reading unintelligible even for an experienced mathematician. As a matter of fact, being a young science, symbolic logic has only established its ideography in recent decades.

On the other hand, contemporary logic has developed as a deductive science. If, in the classical treatises, the norms of correct reasoning were justified one by one, modern logic has taken a different way which, even if inaugurated by Aristotle, had been abandoned: logic is considered today as a deductive science; starting from certain initial axioms and rules, the theorems of logic are linked to one another by this coherence which used to be, since not long ago, specific to mathematical sciences. Here is one of the basic psychological reasons why mathematics is understood with difficulty: the man in the street does not understand mathematics in the first place because he forgets it; the fact of being obliged to remember any theorem, even the simplest one, the one that gave you the least trouble, and therefore was least recorded in your memory, well, there is a disagreeable obligation which constitutes the specific difficulty of the spreading of deductive sciences. The impossibility of just browsing through a work in the field of deductive sciences constitutes a great obstacle in the comprehension and popularity of such a science; we are tempted to say that it is only those who know how to browse through a treatise in that deductive science that can understand it.

We think that some of the ideas highlighted by modern researches in logic are so important that we feel bound to communicate them to an as large as possible circle of readers. Among these ideas, we considered that of a pluralism of logic, which we undertake to introduce in this article, in a form which we try to make as clear as possible.

We should nevertheless notice that the above-mentioned difficulties, which constitute obstacles for the non-professionals – the symbolic and the deductive character of contemporary logic – cannot be overcome in an article of this kind, unless we pay the price of a certain falsification. Like modern mathematics, modern logic has a purely formal character. Ever since its appearance, formal logic has excluded from its preoccupations the research on material truth, but the signification of the term "formal", in the sense it has in mathematical and logic researches, is today more exact than before. In contemporary logic, which is a symbolic one, as we have pointed out, symbols become independent entities, observing strict rules and being totally independent on their interpretation. Manipulating these symbols is pure calculation. Their signification, their translation into rules of active reasoning, their coincidence with the usual grammatical spirit, these are problems outside the area of logic research. Obviously, logic research takes into consideration the suggestions of correct thinking experience, similar to geometry, which, when constituting its concepts, receives suggestions from the experience of physical world: the edges of a ruler or a tense thread suggest the idea of straight line, the sequence of sentences in human spirit suggests the idea of demonstration. But geometry distinguishes itself from experience and its tense threads or its polished surfaces, becoming a purely formal science, whose concepts are defined by its own axioms; similarly, logic is the sublimation of thinking habitude into a purely formal science, based on axioms. If and how geometry can be "applied" to physical world, is a problem of no interest for the geometer; in the same way, knowing whether the concept of "necessity" introduced by certain logic approaches, which we are going to further analyze, coincides with the significance given to this word by "the man in the street", is a problem of mediocre interest for the pure logician. Someone else is supposed to deal with this correspondence, but it is undeniable that that work has to be done by somebody and we have no intention to underestimate this preoccupation. Our intention was to simply point out the very clear distinction between the problems of formal logic and those referring to the interpretation of formal logic.

On the one hand, the characteristic of a deductive discipline will be lost in our exposé; speaking of different logic approaches we are going to introduce them not as deductive systems of theorems, developed in an order of demonstration that starts from certain axioms, but as manners of thinking characterized by these logics. In this way we will give an interpretative, from the outside description to certain deductive and formal theories. Yet, the fundamental problem that we undertake to analyze here, will not be less pointed out in its epistemological aspect.

§ 2

Oftentimes, the ground of the formal logic construction is taken to be the possibility of dividing propositions into true and false ones. No proposition can be both true and false at the same time: this is the principle of contradiction. Any proposition is either true or false and there is no third possibility: it is the principle of excluded middle. We will say that a proposition may have one of the two logical values: true or false. This affirmation constitutes the principle of bivalence. This principle itself allows us to build most of the logic: what is called the logic of propositions – or the calculation of propositions – chapter whose study was started by the stoics, and which has been fully completed by the logicians.

The principle of bivalence justifies a large number of types of argumentations, used quite often. For instance: take proposition p, its negation non p and the negation of this negation: non non p; proposition p may be true or false; if it is false, its negation is true and the negation of the negation is false; if it is true, its negation is false and its double negation is true. So, a proposition is equivalent to its double negation, i.e., they are true and false at the same time. It is the principle of double negation. It proves that, in order to demonstrate a proposition, it suffices to demonstrate that its negation is false, and this constitutes the method of demonstration by *reductio ad absurdum*.

A second example will allow us to justify the principle of the dilemma: if we can draw a conclusion q whether proposition p is true or false, then q is true. As a matter of fact, there are two possibilities: either p is true, or it is false. But we have assumed that if p is true, q is also true; we have also assumed that if p is false q is true, so in this latter case q is true, so q is always true.

The examples given above point out to the nature of the rules of what we call the logic of propositions. We are used, in the spirit of traditional logic, to analyze a sentence in terms of subject, predicate and copula. We do not deny the importance of such an analysis, which we are going to remind to you a little further. But there are certain norms logic of that can be stated independently of this analysis: the rule of the reasoning by *reductio ad absurdum*, the rule of the dilemma, the principle of the double negation are examples of theses of logic in which the internal structure plays no role. The propositions are considered as an indivisible whole and the study refers to the relations and connections between propositions; it is the domain of the logic of propositions.

We can understand the way in which the principle of bivalence intervenes in the logic of sentences if we take the following two examples: in order to prove that a certain rule is true for any proposition, it is enough if it is true irrespective of the values "true" or "false" of any of the propositions it is made of. But the principle of bivalence intervenes when the proposition is analyzed and its subject and predicate are highlighted, as we are going to demonstrate further on.

Here are some immediate remarks: if, in order to demonstrate a theorem of existence we use the reasoning of *reductio ad absurdum*, this existence is only proved as ideal existence, not as possibly constructed existence, as all we have is a demonstration of the affirmation that the inexistence of the object is absurd without any means of approaching the object itself. If I want to make a difference between an existence ideally justified and another one, for which I have given a system of construction, the principle of bivalence will oppose.

The theorems of existence appear quite often in mathematical sciences; generally, the demonstration of a theorem of existence is accompanied by the method of constructing the object whose existence has been demonstrated. For example, the elementary geometry treatises have given different graphic constructions, each of them leading to a theorem that confirms the existence of a figure possessing certain properties: for instance that of a triangle with two angles and a side given. The theorem: there is a number whose second power is a given positive number **a** is accompanied by the rule of extraction of the square root, rule that allows us to find out a number of decimals, which is as large as we want, for the square root of **a**, i.e. allowing us to approximate the square root of **a**.

There are other instances where it may happen that the demonstration of a theorem of existence be done by *reductio ad absurdum*, therefore with the help of the principle of the double negation, without knowing any exact or approximate graphic means of building the object whose existence has been established. Is such a demonstration satisfactory? The problem goes beyond the borders of logic; it is the philosophy of mathematics that deals with it; the two possible extreme attitudes have already been taken: formalism admits these demonstrations unconditioned, intuitionism rejects them; between these two extreme types of philosophy of mathematics there is a whole range of intermediate attitudes.

On the other hand, certain propositions seem not to be subject to the bivalence principle: a judgement regarding the future seems at present neither true nor false. A judgement like the following: the last figure of tomorrow's lottery winning number is 5 is neither true nor false today. Aristotle would not admit for the judgements concerning «contingent future» the principle of bivalence, which he considered to be opposed to freedom, whereas Chrysippos, the determinist is the one who based logic on this very principle[1].

Modern logic will preserve the suggestion: certain judgements, for instance those concerning a contingent future, seem incompatible, unless we violate them, with the framework imposed by the principle of bivalence; as a matter of fact, judgements of this type, very often met in everyday thinking, have not been necessary in building positive sciences.

Finally, even the example given above leads us to a new remark: both in ordinary thinking and in the philosophical one, the true and the false are, to a certain extent, shaded.

The true is sometimes necessary, some other times not necessary or contingent; the false is also sometimes necessarily false and some other times false in a contingent way. We will say that some propositions are necessarily true; a very widespread idea is that mathematical truths are necessary, i.e. that they cannot be untrue; other truths can be contingent; we call contingent what could be different: the result of a game of chance is contingent as, for instance, a die can fall on any of its facets. Upon the necessary future one can make certain predictions; a Romanian poet, Al. Philippide said:

«... you may well try to flee this moment,
It will come, for is bound to ».

We can only make judgements of probability on the contingent future; the logic value of a judgement on the contingent future is the problematic.

Certain philosophers have tried to give criteria of distinction between the necessary and the contingent, between the possible and the impossible. We will not deal with this aspect, but we will make some remarks on the problem. The ideas of problematic, possible, impossible, necessary and contingent have no place within the framework of bivalent logic; they have not been used in building positive sciences and have nothing to do with either mathematics or physics, which have developed within the frame of bivalent logic; on the contrary, everyday thinking as well as philosophical thinking use these modal ideas on a large scale. The pure logician could come up with suggestions for creating a modal logic.

The exposé above makes clear the position of the pure logician, as opposed to the philosopher's, on examining the principle of bivalence. The pure logician will ask himself: how can I build a logic based on the principle of bivalence? He will call this logic a classic one, or better a Chrysippian logic, he will state its theses and try to group the ensemble of Chrysippian logic theses into a deductive branch of science. But, at the same time, the modern logician will raise the problem: can I build non-Chrysippian logic approaches based on other principles than the bivalence one? He will be able to find suggestions in the modal logic.

A simple idea will be to suppose that propositions are divided in three categories: that of true propositions, that of false propositions and that of propositions that are neither true nor false, without being absurd, i.e. they are possible but not necessary, in other words, contingent: the problematic propositions. A proposition could not be at the same time true and false, neither true and problematic, nor false and problematic; there we find a new principle of contradiction. Any proposition is either true, or false or problematic; that is the principle of the excluded fourth. A logic based on these two principles, synthesized in one, the principle of the trivalence, will be called the trivalent Lukasiewiczian logic. The calculation technique for propositions based on the principle of trivalence will be analogous to the one used for the bivalent calculation of propositions. Here is an example: if, supposing that **p** is true and also that **p** is false and finally supposing that **p** is problematic, I can conclude **q**, then **q** is true. This principle of the trilemma can easily be proved following the same steps as above.

Obviously, the trivalent logic will be different from the bivalent one. One important point is the following: we can introduce in the trivalent logic modal ideas, i.e. the ideas of necessary, contingent, possible, impossible. In fact, if a proposition is true and also if it is problematic, it is possible; if it were not possible it would be impossible, so, false. Therefore, the possible is built in a trivalent logic and the false is identified with the impossible. If a proposition is false and also if it is problematic, it is not necessarily true. By calling contingent what can be false – a significance which is close but nor identical to the one used by the scholastic logic – we can see that in the trivalent logic we can build ideas of contingent and necessity, the latter being identified with the idea of truth. We can see that in the trivalent logic the true is identified with the necessary, the false with the impossible, but these two values do not exhaust the domain of all the propositions, which also contains the problematic propositions that are possible but contingent. This is how the trivalent logic makes clear modal ideas, ambiguous in the non-formal thinking.

We can start from this modal distinction, by dividing sentences into two categories: true and false, distinguishing within this framework the necessarily true propositions – in short, necessary – from those which are true, but not necessarily true – on the one hand; on the other hand, the domain of false propositions will be divided into that of absurd propositions and that of false-but-not-absurd ones. This division of propositions into four categories: necessary, absurd, simply true, and simply false constitutes a principle of tetravalence implying a logic which is different from the previous ones: the tetravalent Lukasiewiczian logic. The affirmation that the four classes of propositions have no common element constitutes the principle of contradiction in the tetravalent Lukasiewiczian logic. The affirmation that they exhaust the totality of propositions constitutes the principle of the excluded fifth.

We can go even further. We can introduce a range of shades between the necessary true and necessary false; the problem can be considered as a series of steps: the «more probable» and the «less probable» ones; we can also assume that we have introduced an infinity of possibilities of logic evaluation of propositions. We can thus set up different systems of polyvalent logic, either for a finite or an infinite number of values. Their construction is due to J. Lukasiewicz and that is why we will call them the Lukasiewiczian logic approaches; the Polish school has studied them minutely.

What are the problems raised by a pure logician, problems studied by Lukasiewicz and his students? Once characterized the trivalent, tetravalent, **n**-valent infinite-valent logics for an infinity of values by means of the principle we have stated, a problem of pure formal logic arises: their constitution into deductive systems. It is a matter of stating all the axioms that can found them and all the deductive rules to be employed. It is the problem of the axiomatization of these logics. A second problem, which could be solved in the case of Lukasiewiczian logic, is determining the structure of all the theses of this logic. We only mention the problem of algebrisation. Certain problems of metalogic will be analyzed further.

§ 3

The preceding exposé has led us to a statement which may seem quite strange: there are several different logic approaches. Let's repeat what we have already said: it is not the interpretation of the different logic values that the pure logician is interested in. As a matter of fact, and this is not to be passed over, when discussing the interpretation of word meaning such as: necessary, absurd, contingent, possible, problematic, it may be ambiguous in everyday language, but it is precisely defined by each of the logics mentioned above. Studying the meaning that these modal expressions have for different philosophers and their correspondence with the norms of several value logics, seems to us a problem of high historic interest.

The fact that *several different logics* are now accepted as valid is a establishment, which surpasses the framework of logic research. It is this possibility of accepting the equal validity of several different logics approaches, that contemporary pluralism of logic consists in.

Before analyzing some objections that can be raised, it is not useless to think of the breakthroughs it creates. On the one hand, as a consequence of this remark: mathematics has developed within the norms of Chrysippian logic, we raise the following problem: what would mathematics consist in if we adopt one of the Lukasiewiczian logics, instead of the Chrysippian one?

Yet, the Lukasiewiczian logics approaches described above do not represent the totality of non-Chrysippian logics, i.e. the logics that are not based on the principle of bivalence. Let's mention, among others, Heyting, Johanssen and Kolmogoroff who have built other systems of logic, quite similar to one another, the intuitionist logics, while Lewis has studied the logic of a strict implication. One may, for all non-Chrysippian approaches, raise the problem mentioned above, i.e. of constituting mathematics based on these logic systems. We will notice that classic mathematics is multiplied into a large number of theories. The study of the relations among these theories is in itself a problem that is worth having in view.

But mathematics is not an isolated science; mathematical physics uses mathematics as an instrument for describing nature. Therefore, it is based on the Chrysippian logic. How will this description of the world vary, if, instead of the classical logic, we adopt the different non-Chrysippian logics? Wouldn't it be possible that one of these descriptions be more adequate than the actual descriptions, which have, as we know, great difficulties to overcome? One attempt has been made by Garret Birkhoff and J. von Neumann, others by Paulette Février and Jean Louis Destouches. But in order to answer these questions, first we have to study in detail the non-Chrysippian logic approaches, the relations among them, their structure. There is a widespread idea that opposes this study: it says that logic is one since human thinking is one. But the problem of human thinking as being unique is yet different from the uniqueness of logic. The pluralism of logic is a historical fact. If we have to think over the following problem: how come it is possible to have several different logics, our meditation could only be fruitful if we realized that, in fact, it is possible to have several different logics. So, the problem of the pluralism of logic interests not only the logicians, but also the mathematicians, physicists and philosophers.

§ 4

We find it useless to give another example of ramification of logic in order to highlight the pluralism of logic. Speaking of the non-Chrysippian logics we have not implied the analysis of the sentence with respect to subject-attribute-copula. If for Aristotle and for almost all logicians, until the end of last century, a judgement says something about something else, and therefore it can always be reduced to « S is P », for contemporary logicians this form of predication judgement is just one form that judgements can have, and not even the most frequently employed one. The copula « is » is just one of the numerous copulas that relate the subject to the predicate and it is one of the very numerous relations appearing in human thinking. A general theory of the relations is the work of contemporary logic, which completes Aristotle's logic. To be noted that this completion does not blow up Aristotle's formal system; the theory of relations is one chapter of logic, just as the theory of the syllogism is another, both of them justified, but not opposed to each other. If, in this way, a certain philosophy, based on the very Aristotelian form of judgement, is surpassed, this does no wrong to syllogistics. Moreover, this philosophy is not incompatible with the technique of relational logic, which we can, from a purely philosophical point of view, subordinate to a more general idea of predication. Thus, it would be wrong to call the logic of relations a non-Aristotelian logic. It is a post-Aristotelian chapter of the Aristotelian logic.

Nevertheless, we can ask ourselves: is there anything analogous to the opposition between the Chrysippian logic and the non-Chrysippian logic concerning the very Aristotelian basis of judgement? In order to answer that question we have to analyze how the logicians of last century built the Aristotelian structure of judgement. Let's consider a few judgements: «Socrates is a philosopher». «Cajus is Lucius's brother». «The town of Mărăşeşti lies between Bucharest and Jassy». In all the judgements we deal with one or several precise individuals: Socrates, Cajus, Mărăşeşti, Bucharest, Jassy, which are assigned certain properties, either a predicate, for instance «philosopher», assigned to subject Socrates, or a relation between two subjects or among more, for instance «is the brother of», «lies between», etc. Such sentences are called singular.

Other sentences refer to collectivities or certain general ideas: «all Romanians are true-born poets», «certain Athenians are philosophers». These judgements, called «universal», respectively «particular», may be analyzed as follows. A universal judgement, for instance «any Romanian is a true-born poet» means «for any subject S, if S is Romanian, S is a true-born poet». It can be decomposed into a singular judgement: «if S is Romanian, then S is a true-born poet», having the subject S undetermined and the prefix «for any S», stating that the property is valid for any subject S. This is a quantifying operation, i.e. the assignment of a quantity: a sentence with an undetermined subject S is universally valid. A sentence like: certain Athenians are philosophers means «there is at least one Athenian who is a philosopher», so «there is an S, such as S be at the same time Athenian and a philosopher». It is therefore obtained from the sentence with an undetermined subject S: «S is Athenian and a philosopher» and the prefix «there is an S» stating the property is valid for at least one individual.

In short: the two sentences: the universal and the particular one are made of one sentence with an undetermined subject S and a quantifier, that can be the universal quantifier «for any S», or an existential one «there is an S».

The analysis above is not specific for a predication judgement; it is also valid for the relation judgements. For instance the judgement: «a stupid person will always find someone more stupid to be admired by» may be analyzed as follows: for any S, if S is stupid, there is a T who is more stupid than S and who admires S. This judgement is made of a relation judgement between the two undetermined individuals S and T: «T is more stupid than S and T admires S»; if existentially quantified, it gives a judgement with an undetermined subject S: «there is a T, who is more stupid than S, and who admires S». This last judgement universally quantified gives the judgement we were supposed to analyze.

In this way, by combining the quantifications and the singular judgements with an undetermined subject, we can build sentences having any structure.

The previous analysis, which is due to logic, improves the Aristotelian analysis. It proves that the structure of a judgement is based on two operations: 1) the construction of a singular judgement, relating several subjects, some of them being undetermined, 2) the quantification of a judgement for each of the undetermined subjects. Negation intervenes as an operation of the logic of sentences. The fact that we only need these two quantifiers «for any x…» and «there is an x…» - corresponding to the two quantities a sentence can have in traditional logic: the universal and the particular, is essential and characterizes, in our opinion, Aristotle's logic. These two quantifiers are enough for building classic mathematics, so that we can stress the previous statement, by pointing out that classic mathematics is based on classic logic, and by this we mean a Chrysippian and Aristotelian logic.

Obviously, the Aristotelian analysis of judgements is an act of logic, which is independent from the evaluation of sentences as true or false. That means we can combine the Aristotelian character with the Chrysippian one, but we can also build Aristotelian logic approaches, which are not Chrysippian. Let's give some examples: «there is an S having the property P» and «I deny that no S has the property P». According to Chrysippian logics, the two judgements are equivalent, i.e. they are at the same time true and false, as saying that «I deny that no S has the property P» means that «there is an S, for which it is false that S does not have the property P»; therefore, using the principle of the double negation, we get «there is an S having the property P». According to a non-Chrysippian logic, that does not contain the principle of the double negation, the justification we have given would not be valid. Let's consider, for instance the sentence: «it is impossible that for any S, the property P be impossible». That means that «there is an S for which it is impossible that the property P be impossible». According to a three-value logic, a principle of the double impossibility is valid: if it is impossible for a sentence to be impossible, then this sentence is possible; thanks to this principle, the above-mentioned reasoning leads to the conclusion that «there is an S for which the property P is possible». The equivalence between «it is impossible that for any S, the property P be impossible» and «there is an S for which the property P is possible» constitutes a thesis of the trivalent Aristotelian Lukasiewiczian logic.

In short: we can have Aristotelian non-Chrysippian logic approaches as we can also have a classic logic, which is Chrysippian and Aristotelian. The problem that arises is: can we build a non-Aristotelian logic?

§ 5

It is only at the beginning of this century that arithmetic admitted a non-Aristotelian principle: it is the axiom called the axiom of the choice or Zermello's axiom. In order to set the basis of a certain chapter in modern mathematics, called the theory of sets, Zermelo formulated the axiom: in any non-empty set one can choose an element[2]. The axiom is purely mathematical; it will result in several debates among mathematicians whose analysis cannot find its place here. Hilbert noticed that this axiom can be considered as a purely logic principle and that it can found a logic approach, which is different from Aristotle's. Hilbert states the following principle: given be property P, there exists an individual **a**, such as, if **a** has the property P, then any individual **a** has the property P.

Hilbert draws the attention to the fact that this axiom is not contrary to common thinking. For instance if P is the property of being corruptible, we often think in the following way: if even Aristides is corruptible, then any man is. In this way, we choose an individual, Aristides, who is the least corruptible person and we state that if he, too, is corruptible, then corruptibility is universal. Numerous examples can be found in common thinking. The lover who is convinced that all women are frivolous since the one he had considered the least frivolous is so, does apply Hilbert's axiom, even if the choice may be wrong. The disciple who doubts the whole humanity just because his master has a shortcoming, also applies Hilbert's axiom. Hilbert's axiom is the prototype of the reasoning by «so much the more» or by *a fortiori*.

But we have to point out that Hilbert's axiom does not tell us how we can find the individual, which the reasoning by a fortiori is to be applied to. He states the existence of such an individual and this existence is purely ideal, i.e., not accompanied by any means of construction. Hilbert's axiom is therefore an extreme position of formalism. We notice that, without getting into further details, the fact of accepting or refusing an axiom finally proved to be the result of a philosophical attitude. We consider such a finding to be extremely important for the relations between philosophy and science.

Returning to Hilbert's axiom. It is not essential that for a certain property P one can find an individual **a**, such as, if **a** has the property P then any individual will have the property P, but the fact that this association of the individual **a** with the property P can be made simultaneously for all the properties. To put it in other words, this axiom states the existence of a function, which to every predicate associates an individual: this is Zermelo's epsilon, respectively Zermelo's selective function. Zermelo's axiom is a consequence of Hilbert's axiom.

Can we say that a logic approach based on Hilbert's axiom is non-Aristotelian? Let's notice that in the Aristotelian logic, as well as in Hilbert's, there is a non-constructive principle of existence. We have shown that according to the Aristotelian logic «there is an S having the property P» is similar to «it is false that no S should not have the property P». If we have demonstrated that the sentence «no S has not the property P» is false, we have demonstrated a theorem of existence, without giving any means of construction. The condition of contructiveness of any existence, specific for the intuitionist philosophy of`mathematics is a philosophy that we owe mainly to Brouwer, wich takes an extreme position in the direction of the French school of the beginning of the century and especially to E. Borel's position and is by no means the most suitable to what we used to call Aristotelian judgement, for the very reasons we have presented. A logics of the constructive existence has been partially built by Heyting, as a non-Chrysippian, Aristotelian logics, thus shifting the accent. With a formalist orientation, Hilbert's logics seems a non-Aristotelian logics because the axiom mentioned above implies that we can make the choice of the characteristic element in a totally ideal manner, without even being able to name this individual. It is to be noticed that what we call mathematical idealism has been pushed by Hilbert to an extreme, surpassing thus the medium position, which we used to call Aristotelian.

§ 6

The assertion of the pluralism of logic may frighten those who have not been warned. The one who is keen on stating the unique character of logic may have three different attitudes: on the one hand he may try to prove that it is only the classical, Chrysippian and Aristotelian logic that may lead to no contradiction. It is useless to try it, as one can demonstrate that the other logics as well, lead to no contradiction at all. This is a typical statement for any new discipline: the metalogic. By logic system we mean: the bivalent Aristotelian (Chrysippian) logic of propositions, the trivalent logic of propositions, the bivalent Aristotelian logic of predicates, the trivalent Lukasiewiczian and Aristotelian logic of predicates, the bivalent Hilbertian logic of predicates, the trivalent Lukasiewiczian and Hilbertian logic of predicates, and so on. When asserting that the Chrysippian logic of propositions is not contradictory, I make a judgement having as object an entire deductive discipline. Such a judgement is no longer part of this discipline, but it refers to this logic approach. A discipline has been made up to study the statements on different logics, either separately or in their mutual relations, and this discipline is called metalogic.

One of the essential theorems of the metalogic is this very theorem of the non-contradiction of the logic systems mentioned above[3]. So, trying to demonstrate that it is only the classical logic that is not contradictory can not lead to any result, for the opposite can be closely demonstrated.

The second possible attitude, adopted by several mathematicians with respect to Zermelo's axiom, is to deny the intelligibility of the non-Chrysippian or non-Aristotelian logic approaches. One could ask oneself: how can we distinguish between a true proposition and a necessarily true one? What does it mean that a certain proposition is problematic? Experience gives a single answer to each question: yes or no, and thus imposes the bivalent principle.

To be noticed that the problem of intelligibility surpasses the framework of logic and is at the same time outside the epistemologycal problem. When a mathematician says that he «does not understand» an axiom, he shows that he puts into that axiom more than it expresses. In the same way, he could say that he does not understand how space could be infinite or how two sets, both infinite, can allow a comparison which may lead to the conclusion that the elements of one of the sets are more numerous than those of the other set. We do not deny the interest raised by the problem of intelligibility of mathematics, but we do not think that a problem has to be intelligible in order to be true, in any other sense than «correctly enunciated».

The statement that experience can only give a positive or negative answer to any question, may only lead to the conclusion that propositions referring to direct results of experience are among those that can have only two values. But in a many-valued logics there are propositions that can only have two values, whereas the others can have three values.

The logical pluralism does not exclude the philosophical study of the proposition evaluation criteria, but this research is beyond the framework of formal logic.

1945 (A.B.)

NOTES

1. (Additional note to the Romanian edition). After this paragraph, the original work included the sentence «For the modern logician, the mingling of the idea of freedom and of the principle of determinism does not seem recommendable; he wants to stay within the strict framework of logic, independent from any philosophy».
We find it necessary to be more specific about the meaning that we give to this sentence.

There are propositions that are true irrespective if we join the philosophy of one group or another. For instance the propositions: water boils at 100ºC at the pressure of the atmosphere, the sum of the angles in a triangle is 180º in Euclid's geometry, are valid both within the framework of materialism, Kant's or Plato's philosophy, although the philosophy regarding propositions is different.

The suppressed sentence meant that that is also the situation of propositions in formal logic, mathematized or not.

2. We prefer this variant – false but suggestive – which also has the deficiency of not allowing a distinction between Zermelo's axiom and that of Hilbert (Additional note to the French edition).
3. (Additional note to the French edition). We mean the logic of propositions.