LANGUAGE PROBLEMS IN MATHEMATICS

A number of events in science, particularly significant for its destiny, have brought again the problems of mathematical language to the foreground.

Some of these problems, listed according to their historical order – they are quite recent, actually – are: the theory of relativity and quanta; the conflict between Brouwer's intuitionism and Hilbert's theory of the demonstration; the mathematical valorization of the axiomatic associated to the blooming of the Erlangen Program; the systematic construction of logical algebrae; the logical empiricism of the Vienna School with Carnap's logic syntax and the attempt at creating a unique science; the theory of the geometrical object; the elaboration of a theory of structures and, finally, the deepening of the notion of "chain" – such as it has been created by the calculus of probabilities.

In each of these theories, the problem of language imposes itself as an essential issue, thus re-actualizing the great illusion of a total science, completely mathematized, to which Leibniz had intended to dedicate the years of quietness and retreat that actually never came.

There is no wider spread belief in the identity of science itself, as an organized system of knowledge, with its form of expression and proper language, as in mathematics.

We will make the difference between language, expression and, in a certain sense, history and knowledge, as content of science, only by considering the historic evolution of mathematical doctrines – from geometry to the theory of numbers and of sets – by noticing the endeavor at mathematizing the different fields of physics, by including the scientists' personal experience and considering their individual work not just as a mere circle in the development of science, but as a complete fact, organically existent in itself.

We will never succeed in dismantling the unity of science into form and content, each of them independent, and the whole history of science, the old one, as well as the more recent, which we are going to question in this work, shows that this unity can not be reduced to form, as neither can it be reduced to pure knowledge, form and language being just a mere epi-phenomenon.

Yet, we will notice that in some moments of science the nominalist temptation is looking for its way.

The history of logic, with Hilbert's latest experiments, or the ones with an encyclopedic character, carried out by the Vienna group, has always been trying, using many skills and immense resources, to reduce mathematics, and thus the entire science, to a logic formal system, existing closed in itself, conditioning its real existence outside real human experience.

Relativity has also tried, apparently in a more limited way (but how successfully at the moment!), to mathematize physics, by creating a geometrical model, equivalent to it.

If these attempts had been successful, the image of science would have been unexpectedly poor today: a universal physical science, identical to a mathematical doctrine, and essentially reduced to a system of signs and a few principles. The entire knowledge, with all its virtues, could have been reduced to less than one page.

But we will see, after a short examination of the various attempts, that the history of science, like that of any human act, can not be condensed into pills; that science keeps on living through the vivid work of its creators and that we grasp it in the first place through the language, that gave expression to the scientific thought in all the past experience.

Let's start with *relativity*.

In its restricted form, it appeared in a particularly critical moment for physics.

Maxwell's theory of the electromagnetic phenomena had reached a form that gave full satisfaction in the study of electron dynamics, light phenomena, and electromagnetic phenomena related to bodies at rest. But all the attempts at modifying this theory and adapt it to moving bodies gave no satisfaction. The very simple problem of theoretically studying the influence which translation, the Earth's for instance, exerted upon electrical or optical phenomena, could not get an answer in accordance with experience.

The language of Maxwell's theory proved fundamentally different from that of Newton's mechanics. Each of these theories included the notions of time, space, speed, mass, energy, but with different meanings. As long as physicists did not realize this discrepancy, thinking that the same words were covering the same magnitudes, the merging of these theories in phenomena that had a mixed character, material and electric at the same time, lead only to contradictions and chaos.

The Dutch physicist Lorentz was the first to realize this position of the two doctrines and has tried, in a conservative spirit, to translate into Newton's language what length, time and mass were in the language of electron dynamics.

In spite of having proved inefficient, this cognitive effort lead to the ten-parameter group of transformations (Lorentz) that makes invariant Maxwell's equations as well as the distance in the universe

$$d\sigma^2 = dx^2 + dy^2 + dz^2 - c^2dt^2.$$

If Lorentz had known the geometry of his time, dominated by Poincaré, Klein and by the Erlangen Program, he would have understood that his discovery was the equivalent of a quasi-Euclidean *geometry*, of a four-dimensional world, associated to Maxwell's theory.

One year later, it was Einstein who *understood* the real character of Leibnitz's discovery and proclaimed the mechanical unity of the world, including the mechanics of matter into that of the electron. In order to do that, we had to unify the language, to assign the time, space and mass of the mechanics of matter exactly the meaning they acquired in the dynamics of the electron, interpreted according to the new standpoint we have mentioned.

They reached thus a coherent physical theory for the electromagnetic phenomena of moving bodies.

Obviously, we will still use in mechanics the old Newtonian language, as long as the speeds we are dealing with are at the scale of our everyday experience. We will be using it in the limits of current experience, as people use the dialect of their childhood for the limited use of family and domestic life.

But beyond these borders, we are compelled to use the space and time of the electron, we are obliged to use *a mathematical language* that asks for overwhelming rights in building a physical theory. Time, space, material mass, energy, *measured within a determined experiment*, have only a local and momentary value, associated to the experimenter and the moment the experiment has been carried out. In order to give these experimental results a scientific explanation, we have to *set them* into mathematical formulae in which time, space, mass and energy regain their entire richness of possibilities, depriving thus the experiment of what it had particular and local in it, giving it an expression intelligible for any observer, whenever and in whatever conditions, compatible with Lorentz's transforms.

This is an example of how the function of mathematical language was turned into account in physics.

The geometry associated to physical universe was characterized by the Lorentz group, which, in its turn, corresponds to the calculation law for the distance between two points:

$$(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2 - c^2(t_2 - t_1)^2.$$

Generalized relativity has extended this geometrical vision of the universe, noticing that the presence of a material mass modifies the trajectory of the light ray, therefore the conditions in which the measurements are to be made, and finally the very form that expresses the distance between two points.

In this way, Einstein – by including gravitation in physics – found himself compelled to admit that the representative variety of the space-time universe imposes a calculation rule for the distances between infinitely close points, similar to Riemann's form:

$$ds^2 = g_{ik} dx^i dx^k.$$

This computing rule varies from point to point, according to the distribution of matter in the universe.

In this image, the universe stops to be linear. It is curved, with irregular sinuosities in four dimensions.

It seemed that this time gravitation physics had been entirely structured on geometry, as it had happened with optics. The notion of "four dimensional surface" did serve this target. Einstein's equations offered the possibility of computing its fundamental tensor g_{ik}.

Our hasty intuition thought that from now on, the universe of physics could be studied by a good mathematician on a four dimensional map of Einstein's geometrical universe, with the help of a dictionary meant to translate into physical terms the elements, the lines, the surfaces, their most important properties. What would be significant in geometry and, obviously, independent from the network of coordinates we adopted, would be also significant in physics.

The experimental achievement of a physical object corresponds to a range of operations that lead to the respective geometrical object.

The very correspondence between the two categories of objects is independent on the experimental process, as well as the correlative mathematical scheme. The language of physics is in this conception isomorphic to the geometrical one. In fact, the language of mathematics is not more abstract than that of physics, since the dictionary does not create abstractions.

This turn towards geometry, that revolted many experimental physicists, but which they seemed to be obliged to accept as the unique coherent form and universally valid, the only language in which they could scientifically express their findings, represents an invasion of physics by dominating mathematics. But, in the same time, it is a penetration of a physical standpoint into geometry, of an overwhelming importance for the latter.

For it is only under the influence of this constraint of including and therefore of serving physics, that geometry has cleared up its own existence.

Before 1916, when Einstein's memoir on the bases of general relativity appeared, the idea of "geometrical variety" was very restricted and almost connected with our immediate intuition. A two dimensional variety meant a surface, a piece of fabric, with an as complicated shape as it could be.

The very notion of "nonholonomous surface" imposed by mechanics, upon which the Romanian mathematician Gh. Vrânceanu wrote valuable works later, was kept aside, like a simple curiosity. Other varieties were simply out of question.

That is why, both physicists and geometricians thought in the beginning that the four dimensional geometrical variety, defined by the equations of relativity, was, according to the intuitive model, a common, four dimensional surface.

In reality, physics was interested only in those aspects of the variety that resulted from the knowledge of the rule for measuring distances, i.e. the fundamental metric tensor g_{ik}, or in what was later called " the metric geometry of space". But a surface as well as a space – having any number of dimensions according to that model – also has other properties than the metrical ones; it has projective or other properties. They do not correspond to anything in the physical universe, at least, according to the theory of relativity.

The identification of the universe with a geometric space is therefore impossible under these conditions; *the geometric system of notions* associated to such a space is much more complex than that of physics. A dictionary meant to translate from one language into another and thus to rationalize physics was not possible, because geometry itself was a non-rationalized complex.

It was only after 1917, when Italian mathematician Levi-Civita created the notion of "parallelism" (thus introducing a rule of propagation or transport into a Riemannean variety, which was later on called a "connection") that the proper notion of "Riemannean variety" was elaborated. Giving up the intuitive idea of surface, the Riemannean variety was conceived only with the elements and properties deriving from the connection we are defining, the metrical one, in the case considered by relativity.

Such a variety (that has so little to do – as it was axiomatically defined by É. Cartan – with the old notion of surface) was, indeed, able to constitute a *model* of the physical world, and the axiomatic language of that geometry would have fully answered the aspirations of the physicists, if they had not changed in the meantime and if the progress achieved by geometry had not modified, from a different standpoint, the relations between geometry and physics. Because, while the notion of Riemannean variety was being cleared up, a new light was falling on the rich complex of geometrical possibilities that used to be confusedly mixed on the same surface. Other geometries, other connections have been created by mathematicians, one of the most outstanding of them being, undoubtedly, the French É. Cartan.

The physicist no longer found himself, as it had happened with Einstein in the beginning, in front of a single geometrical model, his only job being to interpret or to translate within this unique model, its physical phenomena.

He had to choose among the different models, the one that suited him best and to justify his choice.

In the meantime, Einstein's first option had proved not valid, particularly as it did not geometrically include electricity.

In search of able to express the physical unity of the world (gravitation – electricity), Weyl, Eddington, Einstein resort to more flexible geometrical varieties than Riemann's primitive one. But, finally, the results did not prove satisfactory and neither did more favorable perspectives open towards achieving a unitary model.

The problem stays open, nevertheless.

If we content ourselves with less, the principle of the geometric model is still valid:

1) a field without matter can be the seat of electromagnetic phenomena;

2) a material field can exist without electricity.

At any rate, a geometric model such as a Riemannean variety, has a local signification. The tensor g_{ik} determined in a certain region, no matter how good the approximation of measurements would be, can not be prolonged too far and, what is worse, neither can we know how far our prolongation may be valid, if it is not made according to an effective experimental investigation.

Any topologic conclusion on the shape or size of the universe, as deduced from local evaluations on this tensor, whether would be deprived of any real support.

All we have to do is think that the metric of the cylinder, cone and plane are exactly the same, and if our physical world had two dimensions, we could not have any possibility to recognize to what extent

PHYSICS = THE RESPECTIVE GEOMETRIC MODEL

If we belong to a plane, a cylinder or a cone. Some experiments of physics would be necessary to inform us on this.

That is why I think that all the evaluations made by numberless astronomers and physicists on the universe in its entireness, starting from the principle of the model, have a feeble basis, for, we can not speak today, in geometry, about such a model in its entireness, but only locally, with the exception of the Euclidean one.

Whatever may be in future the fate of this principle of the model, one thing is certain: its choice out of the multitude of possible models is not imposed by simple logical, necessary criteria, immanent to it or the corresponding physical world.

This choice has been done according to extra-geometrical criteria; it comes from the conveniences of our spirit with respect to physical experience, thus taking us out of the field of language problems, and placing us again within the framework of the experiment and concretely showing us that, however large a language may be, it cannot represent science as such.

The nominative assault aiming at turning science into a pure system of signs, into a logical, pure language, can be considered this time as well, a failure.

The reaction of physical sciences was unexpectedly violent. The new physics is characterized by a manifest anti-nominative tendency. *The double aspect, corpuscular and undulatory,* whose quantic phenomenon is not just a convention, but – according to De Broglie and Bohr – a necessary form of science.

This double aspect, interpreted by the notions of our classical, macroscopic physics, leads to contradictory descriptions.

But nobody is thinking today to leave or alter, for this reason, the notions of this physics, whose edification is related to the entire current human experience. And neither does it seem likely that we could abandon these notions, for any experience, however subtle, however deep into the most delicate phenomena, is recorded on apparata built up by our hands and whose indications can be read by our eyes.

Therefore, our only way of penetrating the field of this microscopic physics is to establish a correspondence between its phenomena and the recordings of our apparata, finally translated into notions belonging to classical physics.

The experimental modalities, focussed on the corpuscular aspect of a phenomenon, lead us to an interpretation that is contradictory, by the very notions that are being used, to the experimental modalities that are approaching the undulatory aspect of the same phenomenon. But that does not mean that an object is at the same time corpuscle and wave, it only means that these two intuitive notions fit only partially the *object* they refer to, and that none of the descriptions is complete, and finally, that both are necessary for its understanding.

We have here an aspect of what Bohr called *complementarity*. Corpuscle and wave are therefore *complementary,* but *not contradictory*, for nobody identifies them with the object, they being only applied in a partial description of the object.

The position of a microscopic object and the quantity of movement, here are other two complementary macroscopic notions. That is why Heisenberg's relations of undetermination have to be understood as establishing this fact of complementarity: if we determine one of these magnitudes, the other one ceases having a determined value.

Once surpassed a certain level of magnitude of elements, level established by Plank's constant, complementarity ceases to function: notions will then valorize their absolute right in the language of physics, resuming their intuitive use.

To summarize, the new physics uses in order to describe microscopic phenomena: *a language*, i.e. that of classical physics, *a principle of correspondence* between the microscopic phenomena on the one hand, and the notions and relations of this language, on the other hand, and finally, *the principle of complementarity,* that determines the use of the precedent one, taking each time into account the concrete form of the experiment we are making.

The enchainment of these principles deprives language of any absolute signification, but preserves its necessary unity for a good understanding among people, by taking refuge on the solid ground of macroscopic physics.

The process of expression of the quantum physics did not stop here; it also looked for another unity, on an abstract, mathematical level, which we are particularly interested in. So much the more, as its achievement was accomplished according to principles totally different from those used by relativist physics.

Relativity aimed at building a geometrical model, isomorphic to physics. The mathematical schemata of the model have a direct and concrete sense.

In the new physics, we meet nothing but abstract mathematical schemata, that have no concrete interpretation in themselves. Such an interpretation is only possible if we specify the manner of experimenting. It is only then that the scheme comes to life and makes work a correspondence between mathematical objects and physical objects.

I will only add that in this theory, statistics has a fundamental role, connected with the fact that the magnitudes are not considered as values in themselves, but – essentially – as results of an experimental process.

That is why we have included the particular schemata of this physics into the scheme of a more general nature constituted of what we, and our collaborator, Gh. Mihoc called a "chain with complete links" and which represents a rule of propagation for a statistical structure:

$$\begin{pmatrix} M \\ \mathrm{d}F(M) \end{pmatrix}.$$

The scheme represented by an operator is in itself a structure, the essential expression of the phenomenon that leads to the achievement of the physical object.

The identity between the object and the structure of the physical phenomenon that leads to the achievement of the object is *a postulate* of the quantic physics. It is true, it acquires a value especially through the mathematical schematism, in which the object and the operating process are equally represented by the operator, but that does not prevent it from being one of its necessary forms.

By resuming an old tradition, physics has found the most adequate language for the quantic phenomena, within the differential processes of the analysis.

Therefore, we will have to take a closer look at the characteristics of this language and its value.

But first, we will briefly show how an investigation on the validity of geometrical doctrines leads us to the same sources: analysis with its numeric elements, and its finite or infinite processes.

The original form, the classical model of any geometry is Euclid's. A system of definitions and conventional postulates appreciates, as everybody knows, its notions, and a system of axioms institutes its fundamental relations. These basic elements, definitions, postulates, axioms are structurally valorized through what we call the Euclidean group of plane transformations.

That is why we can define as objects of this geometry all the forms and invariant relations with respect to the group. So, this is what structurally characterizes the whole geometry.

Inspired by this model, F. Klein conditioned any doctrine constituted as geometry on the following program: a number of facts of the doctrine being given, we determine the amplest group of transformations compatible with them; once the group determined, the doctrine re-systematizes itself according to it, into a real geometry.

Under the influence of these ideas, the non-Euclidean geometries have acquired a final form, joining the respective group of transformations.

É. Cartan's discovery of the group of a Riemannean variety gave a new life to the Erlangen Program, including it into a larger program of a total axiomatization of geometries, which leads to their differentiation into pure disciplines. We are particularly interested in this aspect, as they constitute as many differentiated and structurally different forms of language.

In this way, alongside with the already classical metric geometry, there appeared different projective geometries, whose axioms were given in the works of mathematicians among which, besides Cartan and Veblen, we have to name Barbilian. In the process of constituting a pure geometry, one most important aspect is the criterion that ensures its authenticity, therefore the existence of a model satisfying the system of axioms under consideration and its *uniqueness* – if we do not consider as different all those isomorphic ones.

This criterion consists in the possibility of establishing a numeric interpretation of the elements of geometry, such as they form a *body* (or a *ring*) and generate an *analytical geometry*. We therefore transpose, as Descartes did for Euclid's geometry, any pure geometry into the analytical geometry of a *body* of numbers. The cohesion of this geometry and the analytical game of the functions, equalities and equations, guarantee the cohesion and the logic value of the language of that particular geometry.

For some mathematicians openly, for others, more numerous ones – secretly – it is not only the logical problem of the respective geometry, but also the very doctrine of geometry that have completely been cleared up, exhausted. Except for some technical difficulties, we can state that only when the proper *analytical geometry* has been established, the whole geometry is given.

But as these difficulties are not essential, this position would necessarily lead to the following conclusions: once the axioms given and the analytical geometry constituted, we can build a machine to perform the operations that are characteristic to this science. We would introduce in it the fundamental elements as input data and the machine would perform all by itself, automatically, all the line of relations and theorems that constitute the respective geometry. An axiomatic geometry would therefore be – according to this conception – a machine. And since at the level we are today in the field of computer techniques, the problem seems not to show any effective difficulty, we might think that we are close to the moment when we are going to leave the task of building and even writing geometries to computers.

This is in fact the unavoidable consequence of the logistic standpoint, either under the form Hilbert gave it, or particularly under that given by the Vienna School.

So, in the science of numbers, we have to seek for the trustworthiness of mathematics, the trustworthiness of all its operations, the guarantee that the notions we are dealing with stay pure and identical to themselves along any operation, which is the condition of good functioning for any language, even the non-mathematical one.

This guarantee is obtained – Hilbert used to say – by setting mathematics on solid grounds.

This time, we are no longer talking about systematizing bodies of a special doctrine, but about turning the whole of number science into an axiomatic one, so that besides axioms, postulates and definitions, science can be constituted through the purely formal processes of deductive logic.

Logistics, with Frege, Peano, Russel, Whitehead, Couturat, Wittgenstein had aimed at reaching this ambitious project, demolished by the appearance of a few paradoxes.

Zermelo had tried, following Cantor, an axiomatic of the number sets, but the strange form of his axioms was not met by mathematicians with sympathy. They admit any theory, on the sole condition that axioms should have a clear, precise sense, and lead to a direct and complete spiritual acknowledgement.

The axiom of the choice in particular, introduced by Zermello, raises fundamental objections that divide mathematicians into two decisively opposed groups. The axiom has the following content: a set of elements being given, it is always possible for us to find a procedure through which we can make one determined element correspond to it (and this, whatever the considered set might be, thereby belonging to another, more comprehensively determined set).

The doubtful character of the axiom of the choice comes firstly from the fact that it harms the *existence*, necessarily attached to the definition of the set. By applying it to the main problem of our present preoccupations, namely establishing a language that should more closely correspond to physics, we would find in the very definition that includes all eventual languages, the elements of our choice to make.

This would suppose, as it can be noticed, that we have solved, by purely logical means the very problem of the existence of the language we had been searching for.

The nominalist function of this axiom is obvious and hence the danger of its use in the logic organization of a mathematical language.

Yet, it is true that we can give definitions that may include a principle of choice. But in this case we are not dealing with a descriptive logical definition, but a constructive one, showing *how* each particular element *is obtained*. It is for these sets that Zermelo's axiom is implied, but in reality it no longer has the function of an axiom proper.

For the *descriptively* defined sets, the axiom of the choice has to be taken to pieces, isolating particularly the one referring to the postulation of existence and leaving it aside. But, with this, the axiom ceases having the value it has been primarily assigned.

Without this postulate, Zermelo's logic system falls.

Another one tried to take its place: the system created by Hilbert under the name of "theory of the demonstration", aiming at building some mathematics reduced to *pure demonstration*.

Here are Hilbert's words at the Bologna Conference (1926), showing the necessity of founding a really *trustworthy* mathematics: "a satisfying solution to the *problem of basics* is not possible with these axiomatic procedures (i.e. the procedures of the Zermelo type).

For the axioms used so far include presuppositions having a content (*Inhalt*). If we take as a starting point for the demonstrations axioms having a content, however plausible they may be, mathematics loses its character of *absolute trustworthiness*.

Hilbert's axioms are purely formal, purely logic and, at least in his intention, they fit any content that might be included in mathematics.

On the bases of these axioms, Hilbert considers that he has solved the problem of basics for good, as any mathematical or mathematical-philosophical affirmation is reduced to a precise formula, true or false, which can be rigorously established in theory.

The same ideas are expressed, maybe more concretely, and with such a pure passion, by Herbrand: "the role of mathematics is probably just to furnish reasonings and forms and not to search for the ones to be applied to a certain object".

As the mathematician studying the equation of wave propagation is not supposed to ask himself whether in nature waves do satisfy these equations, similarly, when studying the theory of sets or arithmetic, he should not wonder if the sets or numbers he is intuitively considering verify the hypotheses of the theory under consideration.

The theory of the demonstration leads directly to a *pure machine* that creates out of nothing, by means of simple conventions, theorems, relationships, magnitudes of different orders, equations and integrals1.

This very same machine could furnish geometry the systems of axioms, so the axiomatic geometric doctrines, benefiting this time from an absolute trustworthiness, needing no further guarantee.

The valorization of the formal axioms of the theory was attempted at by Hilbert and his disciples when demonstrating the following principles, whose brief examination is necessary to our exposé: Zermelo's principle, *tertium non datur* (the principle of the excluded middle), the complete induction.

The demonstration of Zermelo's principle, even for the limited cases Hilbert talks about, shows that, in reality, Hilbert's axioms implicitly include a presupposition of existence that can not be demonstrated.

Therefore, this presupposition could not be eliminated from the system of axioms, as their author had wanted to, (and with such a brave pride!). That is why, Hilbert's theory is not a pure theory of the demonstration. It too has to be included in the logistic neo-Platonism of the Vienna School.

The simplest condition for which the principle of the excluded middle has been demonstrated by Herbert is the following: if a sentence is not true for all integer numbers, there is a number for which it is true.

Kronecker had noticed that we are not entitled to define as irreducible a polynomial in x with integer coefficients, if there is no decomposition of the polynomial into another two similar ones.

Hilbert demonstrates "that this definition is perfectly rigorous, so that Krockner's affirmation is not only logically unfit, but also arithmetically inexact".

I am not going to resume the examination of such a demonstration, since this is not the aim of my paper, which is only interested in the critical moments of the problems of language and tries to find out how it got over them.

This criticism, in which Weyl used a few weapons, has been vigorously lead by the Dutch mathematician Brouwer, who has undoubtedly shown that there is a category of propositions, non-contradictory with respect to notorious sentences, about which we can neither say they are true – as Hilbert's scientific logic requires – nor that they are false.

A particularly interesting form of these considerations is the one related to the fact that between a proposition referring to the entireness, when it comes to an infinite set, and a sentence referring to the individual elements of the entireness, there is a game that has to be either verified by experiment, or filled with the help of a demonstration that can not have – once and for all – the universal character of Hilbert's demonstrations, but a particular one, referring to each determined set.

This error of considering that there is no free space for propositions between the entireness and the component parts does not belong to the purely mathematical language only. It is a current error in many sciences, such as the economic ones, or more generally, statistical ones, that use a less rigorous mathematical language and therefore generating logical errors more often than one might expect.

Neither Hilbert and nor his disciples actually did succeed in giving a general demonstration of the principle of the excluded middle. If this had happened, it could not have acquired any other signification than that of showing that the theory of the demonstration essentially contained the same lack of creative power as the principle we are dealing with.

Finally, Hilbert also tried to demonstrate the principle of complete induction.

Poincaré's bright analyses, resumed later by Hadamard, with a particular profoundness, showed that complete induction is a kind of original reasoning, which, whenever applied, really opens a road for creation in mathematics, being impossible to demonstrate it as a general affirmation, starting from axioms of the Hilbert type, if these axioms or the demonstration procedure may not even imply it.

Brouwer gave this principle a large extension, turning it into a real motor of any procedure of mathematical demonstration. Brouwer's intuitionism is based, in the good old Kantian tradition, on the creative role played by time in the mechanism of complete induction that *is accomplished* in time. Mathematical reasoning is, according to this doctrine, a creative march, endlessly developing perspectives that had not existed before their appearance to our spirit, and that have never been exhausted or closed.

Brouwer opposes to tautological mathematics, to logistic, or to the theory of the demonstration, the constructed mathematical disciplines, admitting within their framework only the propositions that are included through the direct constructive process.

Brouwer opposes the conception of mathematics as a simple language, created by the different nominal creations, a more modest sort of mathematics, made up of notions and propositions acquired in science by continuous creation, by the effective effort of the human mind. Each of these sentences carries a name, a sign, the seal of authenticity, which gives it a really human value.

Through this reaction, so violent by the rigor of its framework, that excludes any usage of the principle of the excluded middle, Brouwer's intuitionism places itself near the idealism, actually muh larger, that characterized most of the contemporary mathematicians, from Borel, to Lebesgue and Cartan.

Very close to Brouwer, Borel considers that the only notions, the only definitions that are effectively implied in mathematics are those resulting from a constructive process. As opposed to him, Lebesgue – the first mathematician to have introduced and valorized the axiomatic method in analysis, by defining the integral bearing his name – can not be included among the nominalists, because of the powerful geometric spirit governing his work.

For similar reasons, the Romanian mathematician D. Pompeiu, a positivist like Mach, is still closer to Poincaré, in the great idealist tradition of the 19th century mathematics, which includes – with differences of shades, expressive forms or means of reasoning, more analytic or more synthetic – all the great German mathematicians, as resulting from Bieberbach's famous article.

Resuming the eruption forms of logistics, we will naturally find Vienna as their great center.

The Vienna school has built the most systematic neo-Platonic doctrine. Resuming Bolzano's tradition (for whom the existence of mathematical propositions was established at the moment of their non-contradiction) on the one hand, and that of the logistic, through Wittgenstein's *Tractatus logico-philosophicus* (1922) on the other hand, the thinkers that constituted this "school" adopted a decidedly formal but at the same time realist position towards science, by excluding intuition and time from the knowledge of the world. The manifesto of this school says:

"Scientific description can only include the structure of objects (their formal ordering) and not their essence. There are the structure formulae that join people through language; it is in them that the content of common knowledge is manifested. The subjectively lived qualities as such (red, joy) are nothing but lived acts; they are not knowledge. That is why, physical optics will include nothing but what even a blind man could understand in principle".

"Any act of knowledge is knowledge through its form; form translates the behavior that one is being aware of" (M. Schlick).

In this sense, any science – particularly mathematics – is a language. The latter, in particular, is a tautological language, and it is only the limits of our spirit – as another Viennese Fraenckel says – that prevent us from accepting all its propositions at the same time.

In the radical formulation, given by Otto Neurath, to the violently metaphysical attitude of the "Vienna school", we regain the position which Hilbert took with respect to Zermelo's axiomatic and that of Cantor's other direct disciples.

Scientific enunciations (the existing ones or those relative to a piece of observation or the mathematically-tautological ones) are the only meaningful ones: the propositions that are not included within the framework of science, such as those referring to the reality of things or the theory of knowledge, have a metaphysical character and are deprived of any sense.

Contrary to other less radical ones, Neurath considers even the protocol-type enunciations (the taking down of an experience) as well as those on the propositions of science and which constitute the logical syntax, as belonging to the mass of scientific enunciations, whose only criterion of truth is logical coherence. All these propositions, as relative and for ever apt to improvement as they may be, make up together a system, inside of which, we have to place the order that is necessary for turning it into unitary science – to which several congresses and publications have been dedicated. This order had to come through that integral physicalism, clearly originating from Mach, and it has found an extremely large echo within the world that disliked metaphysical difficulties.

"Any scientific enunciation, is always reduced through a row of tautological enunciations to another, asserting that a certain even took place in a certain place, therefore to a physical enunciation".

Any science, be it a natural science or a moral one, even history, can be expressed in a physical language.

The unity of the language means unitary science, which thereby excludes any meaningless proposition, and therefore not only any metaphysics, but also what is most consistent, any philosophy, as this would require some form of knowledge.

In closely associating logic on the one hand and empiricism on the other hand, the Vienna circle set the final bases of the science about the world, by eliminating all metaphysics, any theory of knowledge, any phenomenology.

In order to achieve this, we need an adequate scientific language, by applying logical analysis and logic criticism to the actual languages.

By eliminating any trace of metaphysics from the language, all the false problems and the glotto logical idols, we prepare the way for a complete merging of logic and experience, thus setting the basis of a new science.

To what extent we can talk about a unique language, by absolutely observing the rules imposed by critical logic, is a problem which can not find a conclusive answer today.

Neurath's idea of resorting in that scope to the current human language from the primitive societies having a pre-metaphysical thinking is obviously a joke. Nevertheless, the attempt at giving mathematical analysis a logically formal structure is a fact, whether it is in the logistic sense, or in Hilbert's more pure approach.

The success was, as we have seen, very feeble. One can only speak of regions of the analysis or the theory of numbers where the theory of the demonstration or the logistic have brought the light of a purifying criticism.

The reversed operation though, was particularly successful. One can speak about the mathematization of logic, achieved by logic algebrae, whose constitution makes obvious and final progress.

I find it worth mentioning that the success of mathematization implied a differentiation of the types of logic, as it happened with geometry, so that we have a number of axiomatic, coherent kinds of logic, real algebrae in their elements and operations. Among the creators of such doctrines we quote

Romanian mathematician Gr. Moisil, who was able to find in these disciplines some models or typical schemes of the scientific reasoning.

A remarkable kind of mathematized logical scheme has been created lately by means of the theory of structures, developed by Ore, Birkhoff and others.

This may seem to point out that the logical process, in its very formal meaning, is not unique, that mathematics is richer in schemes than any of the types of logic represented by the algebrae we have been talking about before, that, any way, the logic level is different from the mathematical one and, for the time being, we can not talk about a language as a consistent reality, if its schematism is to be reduced to the purely logical one. This is the way it is in the science of numbers, it will be much more so in geometry.

We have shown how the system of axioms of a geometry, once established its criterion of authenticity, leads us to an analytical geometry.

Looking for the logic certitude of this language, we reach the above-mentioned conclusions which, referring to the science of numbers in general, also include any analytical geometry and, therefore, any geometry. The geometry-making machine is just an illusion, but an illusion we are not even sorry for.

In order to build a geometry, once the system of axioms given, and its coherence and truthfulness checked, we also need something else than the simple schemes of formal logic. In spite of our good will, no mathematical doctrine seems to have been simple tautology.

With his deep feeling for the values of geometry, D. Barbilian is the first to notice, in *The Axiomatic Critique of the Fundaments of Projective Geometry* that the definition of the categorical system of a geometry, by building its analytical geometry, is not sufficient. This act of defining is complete – according to Barbilian – only when we have succeeded in creating, alongside with the immanent system of algebraic magnitudes, the fundamental group of transformations. This is certainly an important step towards geometric reality, but we have to go even further. Instead of searching for the guarantee of authenticity in analysis and logic, as axiomatic generally does, by stopping at some general propositions, on the threshold of science, let us look, with the help of the fundamental group Barbilian is talking about, for a principle of correspondence with Euclid's geometry. Namely, *not* with the axiomatic scheme, and *not only* with its group scheme, but with the whole doctrine, its propositions, configurations, with the richness of facts and theorems it includes. One can not talk only about a simple logical mechanism that would have produced and selected the notions and theorems that make up the thesaurus of this geometry. It has been thought by the greatest minds of many centuries, it has been invented under different circumstances by whole generations of mathematicians, belonging to all the peoples that collaborated to the history of culture.

A keen sense of values makes the geometrician choose, among the different properties that are presented to him, the most characteristic one. The personal inclinations of expressivity make each mathematician formulate in his own way a theory or even some simple finding.

How many of the theorems discovered along the centuries, are still preserved and how many, even if true, were lost – in spite of having the same value – because of their lacking expressive power, or due to their form of presentation, so that they had to be later on rediscovered in forms meant to ensure their life!

This rich thesaurus, which is Euclid's geometry, is a language indeed, with all the notions, forms of expression and with their own functions.

A syntax of this language is necessary. Its problems surpass Carnap's logical syntax, which has to be unique, somehow above the logic, the only form of philosophy still admitted.

The syntax we are referring to, should embrace those procedures that make possible, among other things, the correspondence between Euclid's geometry, integrally as we conceive it, and any other axiomatic geometry that is being shaped and given reality to, in accordance with the model of this century-old discipline.

This syntax will also clear up the positive contribution of relativism in the particular field of scientific language, where it had the relative proposition valorized, with respect to the predicative, anthropomorphic, one overused by physics, mathematics or even geometry, proper in all the forms of expression – as Bréal was pointing out in his unparalleled volume on semantics.

There is nothing further from the true feeling of the scientist in his contact with science, than the logic automatism which the indefatigable nominalism of so many circles, along so many generations, has lead us to, under the pretext of looking for certainty and truth.

Mathematics is not tautological. An axiomatic-making machine could never build all by itself a geometry or an algebra, as it could not create a living being.

Automatism is not the characteristic of science, even if it uses it in fragments or stages of passage, as humans use any mechanism at hand.

The algorithm can be an instrument of mathematics, but never can it be taken for a way of thinking within it, which many people obviously miss noticing, thus entangling its ways.

Algorithms are comfortable. They have the concrete, material character in a certain sense, of machines. They mechanically divide and systematize the substance of sciences, without our active participation. That is why, once created, they no longer belong to science, except as simple tools and cannot imply for any of the acts they take part in, the responsibility of our science that is thinking, invention, perpetual creation, selection and authentic human value. Neither science in general, nor mathematics in particular are dead, such as the blind formalism of the computer or of the algorithms, they are thinking and life together.

Within the limits of this mathematical language, without identifying itself to science, the algorithm is one of its essential factors.

This language has a grammar of its own, it has autonomous formal rules and a semantic which is very meaningful for the mathematician who discovers in it laws like those of specializing, repartition, radiation, analogy, which add their effects to those of the science proper.

Such an understanding of the problems of science shows us how organically it is included among the disciplines constituted by human spiritual activity.

NOTES

1. I felt a deep emotion when reaching these passages of my discourse, professor Nicolae Iorga, who was not particularly interested in mathematics and neither in its philosophical aspects, left the manuscript he was correcting somewhere at the end of the table I was standing at and talking, and started listening with such attention that I was touched. Note of the author.1